approximately represented by Eq. (1). Two approximations are involved: one of these, the finiteness of M, has already been mentioned; the other is that of replacing a continuous power spectrum by a discrete one (if, of course, the given process has a continuous power spectrum). It is clear that the larger we choose M, the more nearly does the representation, Eq. (1), approximate a normal process. The replacement of the continuous spectrum by a discrete one has a more subtle effect on the validity of the representation. A Gaussian process with a continuous spectral density is known to be ergodic. It can be shown that the process Eq. (1) is nonergodic. This is easily done by considering the criterion that a stationary Gaussian process, $X(t)$, be ergodic[1,4]

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |K_{xx}(\tau)|^2\,d\tau = 0 \tag{18}$$

This condition is not satisfied for a Gaussian process with a discrete power spectral density.

One may then ask, "How can Eq. (1) be used as a representation of an ergodic Gaussian process?" The answer is that so long as one uses ensemble statistics (as opposed to sample statistics), the lack of ergodicity of Eq. (1) is irrelevant provided that the discrete frequencies are chosen close enough together and the time intervals involved are not overly long.

Almost any statistical question that may be asked about a stationary Gaussian process of given power spectral density may be answered to any degree of accuracy required by using Eq. (1) to generate a sequence of time histories and then taking the ratio of the number of records having the required property to the total number of records. For specific problems the number of records required to get meaningful statistical data may be large or small depending on whether the events of interest have a low or high probability of occurrence. Typically, if one studies an event having a probability of occurrence of $\frac{1}{4}$, then perhaps 40 or 50 time histories are sufficient to determine this probability. If, on the other hand, one is concerned with low-probability events (such as first passage times of high crossing levels), then perhaps five or ten thousand records may be required. The question of the feasibility of such an approach is immediately suggested.

Modern, high-speed, digital computers are ideally suited to the routine procedure of generating records from Eq. (1). Standard techniques exist for generating large numbers of independent random numbers from a uniformly distributed population in a very short time (of the order of a few seconds to generate 100,000 numbers). Using such a subprogram it is very simple to write a program that will evaluate Eq. (1) at any specified time intervals and thus produce a digital record of a sample from the given random process. The program may be written to sequentially generate as many records as may be deemed necessary. The analysis of these records for the occurrence or nonoccurrence of events of specified type can be done internally on the computer and a printout of the probability of these events is readily obtained.

In order to test the feasibility of this procedure, a program was written to determine the probability distribution of first passage times for a wide band Gaussian process. The levels studied were fairly low (1.5–2σ). A printout of the relative frequency of occurrence (based on analysis of 500 records) of first passage before time T for 100 values of T was obtained in about 7 min on a CDC 6600 computer. Of course, first passage distributions for higher levels take longer, but if one is faced with the necessity of getting a solution it may well be that computer times of the order of several hours is tolerable. Thus, from an engineering point of view, the first passage problem (as well as many other statistical problems) may be regarded as soluble.

References

[1] Lin, Y. K., *Probabilistic Theory of Structural Dynamics*, McGraw-Hill, New York, 1967.

[2] Rayleigh, J. W. S., *The Theory of Sound*, Vol. 1, Dover, New York, 1945.

[3] Gradshteyn, I. S. and Ryzhik, I. M., *Table of Integrals, Series, and Products*, Academic Press, New York, 1965.

[4] Yaglom, A. M., *An Introduction to the Theory of Stationary Random Functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.

Comparison of Numerical and Asymptotic Expansion Solutions of the Boundary-Layer Equations

DAVID F. ROGERS*

United States Naval Academy, Annapolis, Md.

RECENTLY several authors have presented asymptotic solutions of the boundary-layer equations for limiting cases. In order to assist in establishing the validity of these procedures numerical calculations corresponding to two of these solutions have been carried out. The first considered is that of Brown and Stewartson[1] for the reverse flow solutions of the Falkner-Skan equation when $\beta \to 0^-$ and $f''(0) \to 0^-$. The second is that of Kassoy[2] for the laminar unit Prandtl number compressible boundary layer with blowing when $\beta \to 0^+$ and $f''(0) \to 0^+$.

Brown and Stewartson obtained an asymptotic solution of the Falkner-Skan equation†

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{1}$$

with boundary conditions

$$f(0) = f'(0) = 0 \tag{2a, b}$$

$$f'(\eta \to \infty) \to 1 \tag{2c}$$

for reverse flows ($f''(0) < 0$) in the limit as $\beta \to 0^-$. Here $f(\eta)$ is a nondimensional stream function and the primes denote differentiation with respect to the independent similarity variable η. They showed that in the limit as $\beta \to 0^-$ that the nondimensional shearing stress at the surface is

$$f''(0) = -C(-\beta)^{3/4} \tag{3}$$

where the constant $C = 1.544$ is obtained by numerical integration of an auxiliary equation.

In the present work numerical solutions of Eq. (1) subject to the boundary conditions given in Eq. (2) have been obtained in the limit as $\beta \to 0^-$ and $f''(0) \to 0^-$. The details of the numerical integration scheme are given in Ref. 3. Comparison of the numerical results for $f''(0)$ and the asymptotic solution of Brown and Stewartson[1] is shown in Table 1 and Fig. 1. The relative error given in Table 1 is defined as

$$\text{Relative error} = \frac{f''(0)_{\text{analytical}} - f''(0)_{\text{numerical}}}{f''(0)_{\text{numerical}}}$$

In the limit as $\beta \to 0^-$ these results show excellent agreement between the asymptotic solution of Brown and Stewartson and the present numerical results.

Examination of the nondimensional velocity profiles shows that they exhibit the characteristic reverse flow profile. As $\beta \to 0^-$ the magnitude of the negative velocity decreases while the extent of the reverse flow region increases. Further, small values of shearing stress near the surface for $\beta \to 0^-$ indicate that in this region the effects of the small adverse pressure gradient dominate. However, away from the surface the shearing stress increases signifi-

Received June 16, 1971; revision received January 31, 1972. The partial support of the Office of Naval Research under Project Order PO-0-0057 is acknowledged.

Index category: Boundary Layers and Convective Heat Transfer—Laminar.

* ONR Research Professor; Associate Professor, Aerospace Engineering Department. Associate Fellow AIAA.

† The notation is that of Ref. 1.

Table 1 Solutions of the Falkner-Skan equations as $\beta \to 0^-$

$-\beta$	$-f''(0)^a$ numerical	$-f''(0)$ Eq. (3)	Relative error %	η_{max} numerical
0.02500	0.0743657	0.0970739	30.5	15.0
0.02000	0.0651686	0.0821145	26.0	15.0
0.01500	0.0546716	0.0661783	21.0	18.0
0.01000	0.0423209	0.0488256	15.4	18.0
0.00500	0.0268075	0.0290319	8.3	22.0
0.00250	0.0166347	0.0172624	3.8	22.0
0.00100	0.00861081	0.00868255	0.83	30.0
0.00050	0.00515452	0.00516267	0.16	40.0
0.00025	0.00306895	0.00306975	0.026	50.00

[a] For completeness the numerical results are given to six significant figures but are considered accurate only to within $\pm 5 \times 10^{-5}$.

cantly. Further examination of the numerical results shows that the location of the maximum shearing stress corresponds to the region in which the velocity rapidly increases to the potential flow value. This behavior is analogous to that near the classical flat plate boundary-layer "blow-off" value when there is mass transfer through the surface.

Recently Kassoy[2] using a matched asymptotic expansion technique has investigated the unit Prandtl number compressible similar laminar boundary layer when the mass transfer parameter C is of order one but larger than the classical flat plate "blow-off" value $C_0 = 0.87574\ldots$.[4] He has considered the solution in the limit as $\beta \to 0^+$ and $f''(0) \to 0^+$. In particular Kassoy obtained a solution for ‡

$$f''' + ff'' + \beta(g - f'^2) = 0 \qquad (4)$$

$$g'' + fg' = 0 \qquad (5)$$

subject to the boundary conditions

$$f(0) = -C = \text{const},\ f'(0) = g_w \qquad (6a–c)$$

$$f'(\eta \to \infty) \to 1, \quad g(\eta \to \infty) \to 1 \qquad (6d, e)$$

where $f(\eta)$ is a nondimensional stream function and $g(\eta)$ is a nondimensional enthalpy function. Again the primes denote differentiation with respect to the independent similarity variable η.

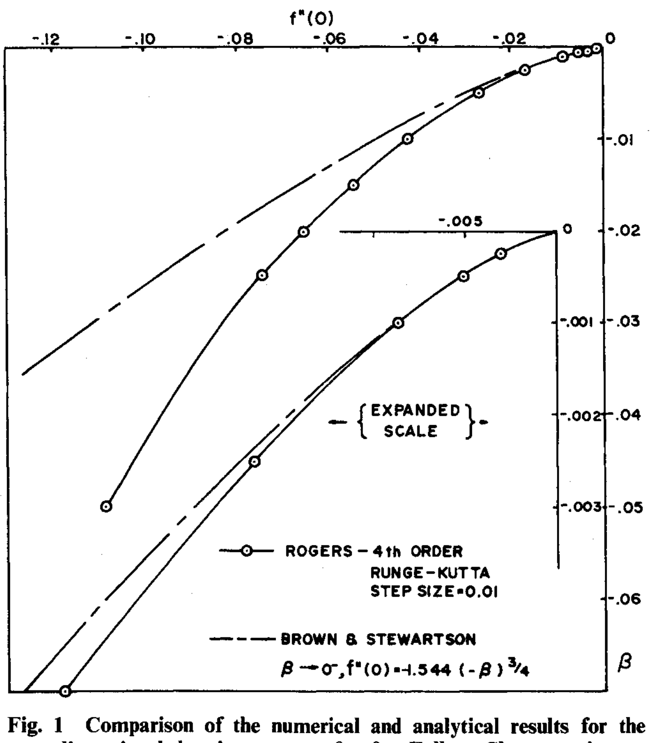

Fig. 1 Comparison of the numerical and analytical results for the nondimensional shearing stress as $\beta \to 0^-$; Falkner-Skan equations.

‡ Here the notation is that of Ref. 5.

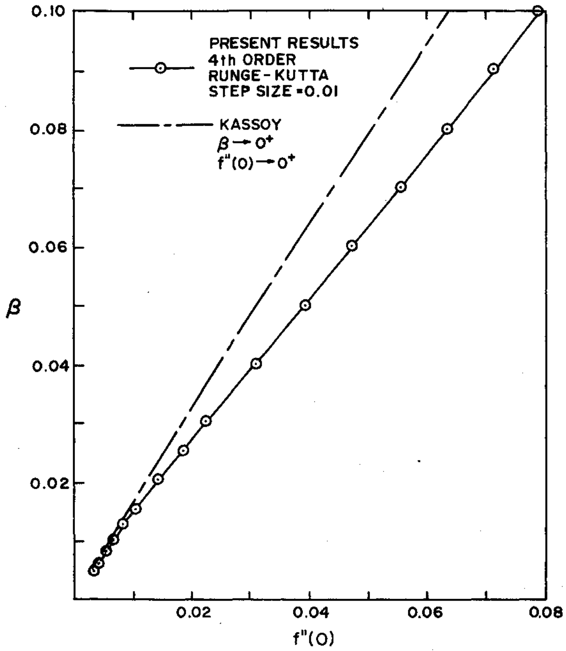

Fig. 2 Comparison of the numerical and analytical results for the nondimensional shearing stress as $\beta \to 0^+$; compressible boundary layer with mass transfer.

Kassoy has shown that the shearing stress at the surface is given by the proportionality

$$f''(0) \sim g_w \beta / C + g_w{}^2 \beta^2 / C^5 \quad \text{if} \quad \ln C/C_0 \gg f''(0) \qquad (7)$$

Further, his solution indicates that in the limit as $\beta \to 0^+$ and $f''(0) \to 0^+$ the fluid layer near the surface is a constant enthalpy field. Hence, there is no heat transfer at the surface.

Numerical solutions of Eqs. (4) and (5) subject to the boundary conditions given in Eq. (6) have been obtained for a wall-to-stagnation temperature ratio $g_w = 0.6$ and a mass transfer parameter $C = 1.0$ for the solution branch corresponding to $\beta \to 0^+$ and $f''(0) \to 0^+$. These results along with those obtained from Eq. (7) are given in Table 2 and Fig. 2. They show that the agreement between the numerical calculations and Eq. (7) for the sharing stress at the surface is satisfactory. For this case the smallest value of β for which solutions have been calculated is 0.005. Solutions can be calculated for smaller values of β. Note

Table 2 Solutions of the compressible boundary-layer equations with mass transfer

$Pr = 1.0,\ g_w = 0.6,\ C = 1.0^a$

β	$f''(0)$ numerical	$g'(0)$ numerical	$f''(0)$ Eq. (7)^b	Relative error %	η_{max} numerical
0.10	0.0787323	0.00792748	0.063600	−19.22	9.0
0.09	0.0711513	0.00714287	0.056916	−20.01	10.0
0.08	0.0634213	0.00632455	0.050304	−20.68	10.0
0.07	0.0555363	0.00547078	0.043764	−21.20	10.0
0.06	0.0474927	0.00458059	0.037296	−21.47	10.0
0.05	0.0392932	0.00365514	0.030900	−21.36	10.0
0.04	0.0309551	0.00270086	0.024576	−20.61	10.0
0.03	0.0225332	0.00173785	0.018324	−18.68	11.0
0.025	0.0183326	0.00126853	0.015225	−16.95	12.5
0.020	0.0141873	0.00082540	0.012144	−14.40	12.5
0.015	0.0101692	0.00043435	0.009081	−10.70	12.5
0.0125	0.00824389	0.000273965	0.007556	−8.34	12.5
0.010	0.00640120	0.000144896	0.006036	−5.71	13.5
0.008	0.00500103	0.0000703856	0.004823	−3.56	14.5
0.006	0.00367428	0.0000241568	0.003613	−1.67	16.0
0.005	0.0030759	0.0000112323	0.003009	−0.94	16.5

[a] For completeness the numerical results are given to six significant figures but are considered accurate only to within $\pm 5 \times 10^{-5}$.

[b] In calculating these results the proportionality has been replaced by an equality.

that although for $C = 1.0$, $\ln C/C_0 = 0.133$ and thus the requirement that $C/C_0 \gg 1$ is not fully satisfied, the agreement between the analytical and numerical results is quite satisfactory. Also note that the numerical solutions show a small but finite nondimensional heat-transfer rate, $g'(0)$ at the surface which approaches zero as $\beta \to 0^+$ and $f''(0) \to 0^+$. In view of these results, Kassoy's asymptotic theory is considered confirmed.

Examination of the nondimensional velocity and shearing stress profiles $f'(\eta)$ and $f''(\eta)$ for the compressible boundary-layer case shows that their behavior is similar to that for the incompressible similar boundary layer. Examination of the numerical results for the nondimensional enthalpy ratio $g(\eta)$ shows that the extent of the region of constant enthalpy near the surface increases as $\beta \to 0^+$ and $f''(0) \to 0^+$. In the limit the constant enthalpy region would become of infinite extent. Consequently no heat transfer would occur at the surface. Further examination of the numerical results shows that the region of maximum heat transfer rate moves away from the surface as $\beta \to 0^+$ and $f'(0) \to 0^+$. These results serve to further confirm the asymptotic solutions obtained by Kassoy.

References

[1] Brown, S. N. and Stewartson, K., "On the Reverse Flow Solutions of the Falkner-Skan Equations," *Mathematika*, Vol. 13, No. 25, June 1966, pp. 1–6.

[2] Kassoy, D. R., "On Laminar Boundary-Layer Blow-off II," *Journal of Fluid Mechanics*, Vol. 48, July 1971, pp. 209–228.

[3] Rogers, D. F., "Further Similar Laminar Flow Solutions," *AIAA Journal*, Vol. 7, No. 5, May 1969, pp. 976–978.

[4] Emmons, H. W. and Leigh, D. C., "Tabulation of the Blasius Function with Blowing and Suction," Combustion Aerodynamics Laboratory Interim TR 9, Nov. 1953, Div. of Applied Sciences, Harvard Univ., Cambridge, Mass.

[5] Moore, F. K., ed., "Theory of Laminar Flows," *High Speed Aerodynamics and Jet Propulsion*, Vol. IV, Princeton University Press, Princeton, N.J., 1964.

Laminar Flamesheet with Hypersonic Viscous Interaction

J. W. Ellinwood* and H. Mirels†

The Aerospace Corporation, El Segundo, Calif.

Introduction

Although analytical studies of laminar flamesheets are available,[1] none has considered the case where the flame is held in a hypersonic stream and the heat release is sufficiently intense to induce a significant pressure increase. Evaluation of this induced pressure is the objective of this Note. For simplicity in other respects, we assume gases with unit Prandtl and Lewis number, linear viscosity-temperature variation, and uniform specific heat for all species. The flow geometry assumed is that of Fig. 1, in which the pressure interaction occurs over a semi-infinite, porous, flat plate with fuel injected through it at a specified angle, and the flamesheet is infinitesimally thin. Only self-similar flows are considered. This requires an injection distribution diminishing downstream as $x^{-1/2}$ or $x^{-3/4}$, for weak or strong interaction cases, respectively, and an isothermal wall.

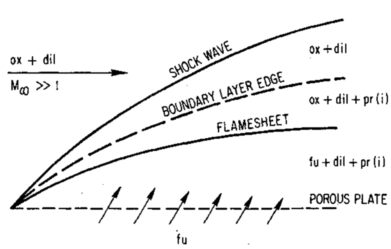

Fig. 1 Self-similar flow model describing pressure interaction between hypersonic flow and laminar flamesheet.

Received August 16, 1971. This work was supported by the U.S. Air Force under Contract F04701-71-C-0172.

Index categories: Jets, Wakes, and Viscid-Inviscid Flow Interactions; Combustion in Gases.

* Member of the Technical Staff. Associate Fellow AIAA.

† Head, Aerodynamic and Heat Transfer Department, Laboratory Operations. Associate Fellow AIAA.

Analysis

The self-similar equations of motion for the boundary layer are

$$f''' + ff'' + \beta(g - f'^2) = 0 \quad \text{(momentum)} \tag{1}$$

$$g'' + fg' = 0 \quad \text{(energy)} \tag{2}$$

$$p = \gamma^{-1}(\gamma - 1)\rho h \quad \text{(state)} \tag{3}$$

$$C_i'' + fC_i' = 0 \quad \text{(species)} \tag{4}$$

where

$$f = (2\xi)^{-1/2}\psi, \qquad g = h/H_e \tag{5a, b}$$

$$p = p_\infty P, \qquad \Sigma_i C_i = 1 \tag{6a, b}$$

$$\eta = u_e(2\xi)^{-1/2}\int_0^y \rho\, dy, \qquad \xi = \int_0^x \rho_e \mu_e u_e\, dx \tag{7a, b}$$

$$\beta = (2\xi/u_e)(du_e/d\xi)H_e/h_e \tag{8}$$

Primed quantities denote differentiation with respect to the lone independent variable η. The symbols p, γ, ρ, h, ψ, H, μ, and u represent the gas pressure, specific heat ratio, density, enthalpy, stream function, stagnation enthalpy, viscosity, and x component of velocity, respectively. C_i is the mass fraction of species i. These equations apply between the plate and the flamesheet and between the flamesheet and the outer edge of the boundary layer. Boundary conditions must be applied at the plate, the flamesheet, and the outer edge. At the plate ($y = 0$)

$$-f_w = N\left(2x^{-1}\int_0^x P\, dx\right)^{1/2} \Big/ P \tag{9}$$

$$f_w' = g_w N\bar{\chi}/(M_\infty P \tan\theta) \tag{10}$$

$$C_{i,w}' + f_w C_{i,w} = 0 \qquad (i \neq \text{fu}) \tag{11a}$$

$$C_{\text{fu},w}' + f_w C_{\text{fu},w} = f_w \tag{11b}$$

and g_w is given, where

$$N = \rho_w v_w(\rho_\infty u_\infty)^{-1} Re_\infty^{\ 1/2} \tag{12}$$

$$Re_\infty = \rho_\infty u_\infty x/\mu_\infty, \qquad \bar{\chi} = \tfrac{1}{2}(\gamma - 1)M_\infty^{\ 3} Re_\infty^{\ -1/2} \tag{13a, b}$$

and v_w is the transverse component of velocity at the surface and M is the Mach number. At the flamesheet, f, f', f'', g, C_{dil}, and $C_{\text{pr}(i)}$ are continuous, where subscripts fu, dil, pr (i), and later ox denote fuel, diluent, the ith product, and oxidizer, respectively (see Fig. 1). In addition,

$$g'(\eta_*^{\ +}) = g'(\eta_*^{\ -}) - g_C C_{\text{ox}}'(\eta_*^{\ +})/C_{\text{ox}}(\infty) \tag{14}$$

$$C_{\text{ox}}(\eta_*^{\ +}) = C_{\text{fu}}(\eta_*^{\ -}) = 0 \tag{15}$$

$$v_{\text{ox}} W_{\text{ox}} C_{\text{fu}}'(\eta_*^{\ +}) + v_{\text{fu}} W_{\text{fu}} C_{\text{ox}}'(\eta_*^{\ -}) = 0 \tag{16}$$

where W is the molecular weight and v the stoichiometric coefficients of the following chemical reaction:

$$v_{\text{fu}}[\text{fu}] + v_{\text{ox}}[\text{ox}] \to \Sigma_i v_i[\text{pr}(i)] + \Delta H \tag{17}$$

$$g_C = \Delta H C_{\text{ox}}(\infty)/(v_{\text{ox}} W_{\text{ox}} H_\infty) \tag{18}$$

Physically, g_C is the ratio of combustion energy per unit free-stream mass to freestream stagnation enthalpy per unit mass (H_∞). At the outer edge ($\eta \to \infty$), $C_{\text{ox}}(\infty)$ is a specified constant and

$$f'(\infty) = g(\infty) = 1 \tag{19}$$

$$C_{\text{pr}(i)}(\infty) = 0 \tag{20}$$

$$P = \begin{cases} 1 + (\gamma/2)M_\infty \delta^*/x & \text{(weak interaction)} \\ (kM_\infty \delta^*/x)^2 & \text{(strong interaction)} \end{cases} \tag{21a, b}$$